Linear Algebra

UNIT-5

SINGULAR VALUE DECOMPOSITION

feedback/corrections: vibha@pesu.pes.edu

VIBHA MASTI

POSITIVE DEFINITENESS

- A symmetric matrix $A \Rightarrow A = A^T$ (real symmetric matrix)
- Eigenvalues of symmetric matrices are always real and eigenvectors can be chosen to be perpendicular
- For symmetric cases, eigenvector matrix S is orthogonal (orthonormal columns)

$$A = S \Lambda S^{-1} = S \Lambda S^T = Q \Lambda Q^T$$

spectrum

- Symmetric matrix: signs of pivots are signs of eigenvalues

POSITIVE DEFINITE MATRIX

- A symmetric real square matrix A is a positive definite matrix if

1. All eigenvalues are positive
2. All pivots are positive
3. * $x^T A x > 0$ (except origin)
4. All subdeterminants are positive (upper left)

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & 5 & 4 \\ 1 & 4 & 2 \end{bmatrix} \rightarrow \begin{vmatrix} 1 \end{vmatrix} > 0, \quad \begin{vmatrix} 1 & 4 \\ 4 & 5 \end{vmatrix} > 0, \quad \begin{vmatrix} 1 & 4 & 1 \\ 4 & 5 & 4 \\ 1 & 4 & 2 \end{vmatrix} > 0$$

if all 3 are true, matrix is positive definite

POSITIVE DEFINITE MATRICES & LEAST SQUARES

- A symmetric matrix A is positive definite iff there is a matrix R with independent columns such that

$$A = R^T R$$

- LDU decomposition Cholesky Decomposition

$$A = LDU = LDL^T = R^T R$$

$$R^T R = L\sqrt{D}\,\sqrt{D}\,L^T$$

$$\boxed{R = \sqrt{D}\,L^T} \qquad R^T = \sqrt{D}\,L \qquad \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{bmatrix}\begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{bmatrix}$$

- Diagonalisation

$$A = Q\Lambda Q^T = R^T R$$

$$\boxed{R = \sqrt{\Lambda}\,Q^T}$$

Semi Definite MATRICES

- Subdeterminants can be 0 (singular)
- Eigenvalues ≥ 0
- Pivot test not relevant (insufficient pivots)

- $x^T A x \geq 0$

- $A = R^T R$; R can have dependent columns

Q1. Find the symmetric matrix A which gives the quadratic form

$$3x^2 - 2y^2 + 8xy - 5yz + xz + z^2$$

Assume $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $\quad x^T A x > 0$

$$x^T A x = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} ax + hy + gz & hx + by + fz & gx + fy + cz \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= ax^2 + hxy + gzx + hxy + by^2 + fzy + gxz + fyz + cz^2$$

$$= ax^2 + 2hxy + 2gxz + by^2 + 2fzy + cz^2$$

$a = 3$ $\quad 2h = 8$ $\quad 2g = 1$ $\quad 2f = -5$

$b = -2$ $\quad h = 4$ $\quad g = 1/2$ $\quad f = -5/2$

$c = 1$

$$A = \begin{bmatrix} 3 & 4 & 1/2 \\ 4 & -2 & -5/2 \\ 1/2 & -5/2 & 1 \end{bmatrix}$$

Q2. Find the symmetric matrix A which gives the quadratic form

$$5x_1^2+3x_2^2+2x_3^2-x_1x_2+8x_2x_3+0x_1x_3$$

$$A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$eq = ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

$a=5$ $\quad$ $2f=8$ $\quad$ $2g=0$ $\quad$ $2h=-1$

$b=3$ $\quad$ $f=4$ $\quad$ $g=0$ $\quad$ $h=-1/2$

$c=2$

$$A=\begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

Q3. Find the symmetric matrix A which gives the quadratic form

$$8x_1^2+7x_2^2-3x_3^2-6x_1x_2+4x_1x_3-2x_2x_3$$

$$A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$eq: ax_1^2+bx_2^2+cx_3^2+2fx_2x_3+2gx_3x_1+2hx_1x_2$$

$a=8$ $\quad 2f=-2$ $\quad 2g=4$ $\quad 2h=-6$

$b=7$ $\quad f=-1$ $\quad g=2$ $\quad h=-3$

$c=-3$

$$A=\begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

Q4. Find the symmetric matrix A which gives the quadratic form

$$10x_1^2 - 6x_1x_2 - 3x_2^2$$

$$A=\begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

$$x^T A x$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} ax+hy & hx+by \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= ax^2 + 2hxy + by^2$$

$a=10$ $\quad 2h=-6$

$b=-3$ $\quad h=-3$

$$A=\begin{bmatrix} 10 & -3 \\ -3 & -3 \end{bmatrix}$$

Q5. Compute the quadratic form x^TAx for A

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 4x+3y & 3x+2y+z & y+z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= 4x^2+3xy + 3xy+2y^2+ yz + yz+z^2$$

$$= 4x^2+ 2y^2+ z^2+ 6xy + 2yz$$

Q6. Check if A is positive definite or positive semi definite

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

1. All eigenvalues pos
2. All pivots positive
3. $x^TAx > 0$
4. Subdeterminants >0

1. pivots

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 1/2 R_1]{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

not positive

value of 3rd pivot = 0

$\therefore$ cannot be positive definite

may be positive semi definite

2. Eigenvalues: $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}$$

$$-\lambda^3 + 6\lambda^2 - (3+3+3)\lambda + 2(3) + (-2-1) - 1(1+2)$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$-\lambda(\lambda^2 - 6\lambda + 9) = 0$$
$$-\lambda(\lambda-3)(\lambda-3) = 0$$

$\lambda = 0$ and $\lambda = 3$

3. Subdeterminants

$D_1 = 2 \qquad D_2 = 3 \qquad D_3 = 0$

4. $x^T A x > 0$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x-y-z & -x+2y-z & -x-y+2z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$2x^2 - yx - zx \quad -xy + 2y^2 - yz \quad -xz - yz + 2z^2$$

$$(x-y)^2 + (y-z)^2 + (z-x)^2 > 0$$

if $x = y = z$, $\quad x^T A x = 0$

$\therefore$ semi definite matrix

Q7. Check if positive definite

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

1. Pivots

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2/3 R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

2. Eigenvalues

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - (3+4+3)\lambda + 2(3) + 1(-2)$$

$$-\lambda^3 + 6\lambda^2 - 10\lambda + 4 = 0$$

$$\lambda_1 = 2-\sqrt{2}, \quad \lambda_2 = 2+\sqrt{2}, \quad \lambda_3 = 2$$

3. Subdeterminants

$$D_1 = 2 \qquad D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \qquad D_3 = 4$$

4. $x^T A x > 0$ $\quad\quad A = LDL^T$

$$(x^T L)\, D\, (L^T x)$$
$$(L^T x)^T\, D\, (L^T x)$$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix} \xrightarrow[\substack{R_2 \to 2/3 R_2 \\ R_3 \to 3/4 R_3}]{R_1 \to 1/2 R_1} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$$(L^T x) = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y/2 \\ y - 2/3 z \\ z \end{bmatrix}$$

$$(L^T x)^T\, D\, (L^T x)$$

$$= \begin{bmatrix} x - y/2 & y - 2/3 z & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} x - y/2 \\ y - 2/3 z \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x - y/2 & y - 2/3 z & z \end{bmatrix} \begin{bmatrix} 2(x - y/2) \\ 3/2\,(y - 2/3) \\ 4/3\,(z) \end{bmatrix}$$

$$2(x-y/2)^2 + 3/2\,(y-2/3\,z)^2 + 4/3\,(z)^2 > 0$$

$\therefore$ positive definite

Q8. Check for positive definiteness and semi definiteness

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+C \end{bmatrix}$$

1. Pivot

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+C \end{bmatrix} \xrightarrow[R_3 \to R_3 + 1/2 R_1]{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2+C \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & C \end{bmatrix}$$

for $c < 0$, not semi definite

for $c \leq 0$, not positive definite

2. Subdeterminants

$D_1 = 2 \qquad D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$D_3 = 2(4+2C-1) + 1(-2-C-1) - 1(1+2)$

$= 6 + 4C - 3 - C - 3$

$= 3C$

not semi definite if $c < 0$

not positive definite if $c \leq 0$

3. $x^T A x > 0$

$$[x \ y \ z] \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+C \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[x \ y \ z] \begin{bmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + (2+c)z \end{bmatrix}$$

$= 2x^2 - xy - xz - xy + 2y^2 - yz - xz - yz + (2+c)z^2$

$= 2x^2 + 2y^2 + (2+c)z^2 - 2xy - 2yz - 2xz$

$= (x-y)^2 + cz^2 + (x-z)^2 + (y-z)^2 > 0$

if $c > 0$, positive definite

if $c \geq 0$, semi definite

4. Eigenvalues $\quad |A - \lambda I| = 0$

$$-\lambda^3 + (6+c)\lambda^2 - (4+2c-1+4+2c-1+3)\lambda + 3c = 0$$

$$-\lambda^3 + (6+c)\lambda^2 - (4c+9)\lambda + 3c = 0$$

$$-(\lambda-3)(-(3+c)\lambda + c + \lambda^2)$$

$\lambda_1 = 3$

$\lambda_2 = \frac{1}{2}\left(\sqrt{c^2+2c+9} + c + 3\right)$

$\lambda_3 = \frac{1}{2}\left(-\sqrt{c^2+2c+9} + c + 3\right)$

if $c > 0$, $\lambda_3 > 0$

$\therefore$ positive definite if $c > 0$
semi definite if $c \geq 0$

Q9. Test for positive definiteness and write the corresponding quadratic forms

(i) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

(i) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

1. Pivot

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}$$

negative

$\therefore$ not +ve def

(ii) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

1. Pivot

$\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$ $\therefore$ not positive definite

(iii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ 1. Subdeterminants: $D_1 = 1$

$D_2 = 0$

$\therefore$ not +ve def

(iv) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

1. Pivot

$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$P_1 = |A_1| = 2$

$P_2 = \dfrac{|A_2|}{|A_1|} = \dfrac{4-1}{P_1} = \dfrac{3}{2}$

$P_3 = \dfrac{|A_3|}{|A_2|} = \dfrac{2(3)+1(-2-1)-1(1+2)}{3/2}$

$= 0$

$\therefore$ not +ve definite

(v) $\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

1. Pivot

$P_1 = |A_1| = 3$

$P_2 = \dfrac{|A_2|}{|A_1|} = \dfrac{2}{3}$

$P_3 = \dfrac{|A_3|}{|A_2|} = \dfrac{3(-2)-2(2)}{2} = -5$

$\therefore$ not +ve definite

(vi) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \therefore A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

1. Eigenvalues

$$|B| = \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & 1 \\ 2 & 1 & -\lambda \end{vmatrix}$$

$$-\lambda^3 + 0\lambda^2 - (-1-4-1)\lambda + 0 - 1(-2) + 2 = 0$$

$$-\lambda^3 + 6\lambda + 4 = 0$$

$$\lambda = -2 \longrightarrow \lambda_B = 4$$
$$\lambda = 1+\sqrt{3} \longrightarrow \lambda_B = 4+2\sqrt{3}$$
$$\lambda = 1-\sqrt{3} \longrightarrow \lambda_B = 4-2\sqrt{3}$$

2. Pivots

$P_1 = 0 \longrightarrow$ not +ve def

Singular Value Decomposition

- Any $m \times n$ matrix A can be factored into

$$A = \underbrace{U}_{\text{orthogonal}} \; \overbrace{\Sigma}^{\text{eigenvalues (diagonal)}} \; \underbrace{V^T}_{\text{orthogonal}}$$

- The columns of $U_{m \times m}$ are eigenvectors of $\overbrace{AA^T}^{\text{symmetric}}$

$$AA^T = U\Sigma \boxed{V^T V} \Sigma^T U^T = U \Sigma \Sigma^T U^T = U \Sigma \Sigma^T U^T$$

- The columns of $V_{n \times n}$ are eigenvectors of $A^T A$

$$A^T A = V \Sigma^T \boxed{U^T U} \Sigma V^T = V \Sigma^T \Sigma V^T = V \Sigma^T \Sigma V^T$$

- For positive definite matrices, Σ is Λ

- U & V give orthonormal bases for four fundamental subspaces of A
  - Column space: first r columns of U
  - Row space: first r columns of V
  - Left null space: last $m-r$ columns of U
  - Null space: last $n-r$ columns of V

- For detailed explanation on cases, read LA unit 5- UE18.pdf, pg 4

$$A_{m \times n} = \underbrace{U_{m \times m}}_{\text{orthonormal}} \; \underbrace{\Sigma_{m \times n}}_{r \text{ singular values on diagonal}} \; \underbrace{V^T_{n \times n}}_{\text{orthonormal}}$$

COMPUTE SVD for a MATRIX

Case 1: Matrix A is a short matrix (eg: 2x3)

1. Find AA^T (2x2)

2. Find eigenvalues of AA^T : λ_1 & λ_2

3. Find corresponding eigenvectors x_1 & x_2

4. Normalise them to get u_1 & u_2 such that

$$U = [u_1 \ \ u_2]_{2\times 2}$$

5. Find singular values $\sigma_1 = \sqrt{\lambda_1}$ & $\sigma_2 = \sqrt{\lambda_2}$ such that

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix}_{2\times 3} \qquad (\lambda_1 > \lambda_2)$$

6. No need to find A^TA. Eigenvalues of A^TA are λ_1, λ_2 and 0

7. Use formula $v_i = \dfrac{A^T u_i}{\sigma_i}$ or $v_i^T = \dfrac{u_i^T A}{\sigma_i}$ and find v_1 & v_2

8. Find v_3 using orthogonality (v_3 is orthogonal to v_1 & v_2)

$$\begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

V_3 = normalised null space

9. Matrix $V = [v_1 \ \ v_2 \ \ v_3]_{3\times 3}$

10. Write $A = U \Sigma V^T$

Eg: SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times 3}$

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Eigenvectors of AA^T are columns of U

Eigenvalues: $\begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 34\lambda + 225 = 0$$
$$(\lambda-25)(\lambda-9) = 0 \Rightarrow \lambda_1 = 25 \quad \lambda_2 = 9$$

(i) $\lambda = 25$

$$\begin{bmatrix} 17-25 & 8 \\ 8 & 17-25 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -8 & 8 \\ 0 & 0 \end{bmatrix}$$

$$-8x + 8y = 0 \Rightarrow x = y$$

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(ii) $\lambda = 9$

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2\times 2}$$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3}$$

$$v_1 = \frac{A^T u_1}{5} = \frac{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{5} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$v_2 = \frac{A^T u_2}{3} = \frac{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{3} = \begin{bmatrix} -1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for v_3 (Null space)

$$v_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 0 & -4/3\sqrt{2} & 1/3 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

Case 2: Matrix A is a tall matrix (eg: 3x2)

1. Find A^TA (2x2)

2. Find eigenvalues of A^TA : λ_1 & λ_2

3. Find corresponding eigenvectors x_1 & x_2

4. Normalise them to get v_1 & v_2 such that

$$V = [v_1 \quad v_2]_{2\times 2}$$

5. Find singular values $\sigma_1 = \sqrt{\lambda_1}$ & $\sigma_2 = \sqrt{\lambda_2}$ such that

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix}_{3\times 2} \qquad (\lambda_1 > \lambda_2)$$

6. No need to find AA^T. Eigenvalues of AA^T are λ_1, λ_2 and 0

7. Use formula $u_i = \dfrac{Av_i}{\sigma_i}$ and find u_1 & u_2

8. Find u_3 using orthogonality

9. Find $U = [u_1 \; u_2 \; u_3]$

10. Write $A = U \Sigma V^T$

Case 3: Matrix A is a square matrix

1. Find if $A = A^T$ (symmetric) and then check if A is positive definite

2. If A is positive definite, SVD is same as diagonalisation

$$A = U \Sigma V^T = Q \Lambda Q^T$$

eigenvector matrix (orthonormal)

eg: $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

(using Gram-Schmidt process)

3. If A is not positive definite, find eigenvectors of AA^T (columns of U) and A^TA (columns of V).

4. Follow the steps to find U, Σ and V from case 1 & case 2

Q10. Find SVD for $A = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$

large $\longrightarrow$ small eigenvalues

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T \Sigma V^T$$

$$A^T A = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix} = V \Sigma^T \Sigma V^T$$

eigenvectors $A^T A \longrightarrow$ columns of V

$$|A^T A - \lambda I| = 0$$

$$\begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 100\lambda + 1924 - 324 = 0$$

$$\lambda^2 - 100\lambda + 1600 = 0$$

$$(\lambda - 80)(\lambda - 20) = 0$$

$$\lambda_1 = 20 \qquad \lambda_2 = 80$$

eigenvectors

(i) $\lambda = 20$

$$(A^TA - 20I)x = 0$$

$$\begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 6 & 18 \\ 0 & 0 \end{bmatrix}$$

let $y = k$

$$6x + 18k = 0$$
$$x = -3k$$

$$\vec{x} = \left\{ \begin{bmatrix} -3k \\ k \end{bmatrix}, k \in R \right\} = \left\{ k\begin{bmatrix} -3 \\ 1 \end{bmatrix}, k \in R \right\}$$

unit vector $\vec{x} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

(ii) $\lambda = 80$

$$\begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 + 1/3 R_1$$

$$\begin{bmatrix} -54 & 18 \\ 0 & 0 \end{bmatrix}$$ let $y = k$

$-54x + 18k = 0$

$x = \frac{1}{3}k$

$\vec{x} = \left\{ k \begin{bmatrix} 1 \\ 3 \end{bmatrix}, k \in R \right\}$

unit vector $= \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

$AV = U\Sigma$

$\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = U\Sigma$

$= \begin{bmatrix} -10/\sqrt{10} & 20/\sqrt{10} \\ 10/\sqrt{10} & 20/\sqrt{10} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -10 & 20 \\ 10 & 20 \end{bmatrix}$

$= \sqrt{10} \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$

$= \sqrt{10} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 2\sqrt{20} \end{bmatrix}$

$$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 2\sqrt{20} \end{bmatrix}$$

$$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

Q11. Find SVD for $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

$$AV = U\Sigma$$

symmetric positive / semi definite

$$\underline{A^TA} = V\Sigma\Sigma^T V^T \qquad S\Lambda S^{-1} = Q\Lambda Q^T$$

eigenvectors of A^TA : V

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16+64 & 12+48 \\ 12+48 & 9+36 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 80-\lambda & 60 \\ 60 & 45-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 125\lambda + 3600 - 3600 = 0$$

$$\lambda(\lambda - 125) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 125$$

eigenvectors

(i) $\lambda = 0$

$$\begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \rightarrow R_2 - 3/4 R_1$$

$$\begin{bmatrix} 80 & 60 \\ 0 & 0 \end{bmatrix} \qquad \text{let } x_2 = k$$

$$80x_1 + 60k = 0$$

$$x_1 = -\frac{3}{4}k$$

$$x = \left\{ k \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \; k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

(ii) $\lambda = 125$

$$\begin{bmatrix} -45 & 60 \\ 60 & -80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 + 4/3 R_1$$

$$\begin{bmatrix} -45 & 60 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = k$

$$-45x_1 + 60k = 0$$

$$x_1 = \frac{4}{3}k$$

$$x = \left\{ k \begin{bmatrix} 4 \\ 3 \end{bmatrix}, k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$V = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$AV = U\Sigma$$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{5} \end{bmatrix}$$

COVARIANCE MATRIX

- Principle component analysis
- Sample mean : M

Mean Deviation Form

Let $[x_1 \ x_2 \ \dots \ x_n]$ be a matrix

$$M = \frac{1}{N}(x_1 + x_2 + \dots + x_n)$$

$$\hat{x}_k = x_k - M$$

$$B = [\hat{x}_1 \ \hat{x}_2 \ \dots \ \hat{x}_n]$$

$$S = \frac{1}{N-1} \underbrace{BB^T}$$

positive semi definite

Q12. Find covariance of the following. Compute sample mean and the covariance matrix

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} \quad x_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} \quad x_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

Sample mean

$$M = \frac{1}{4}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} \right)$$

$N = 4$

$$= \frac{1}{4}\left(\begin{bmatrix} 20 \\ 16 \\ 20 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

$$M = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$ sample mean

Subtract M from all vectors

$$B = \begin{bmatrix} 1-5 & 4-5 & 7-5 & 8-5 \\ 2-4 & 2-4 & 8-4 & 4-4 \\ 1-5 & 13-5 & 1-5 & 5-5 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$S = \frac{1}{4-1} BB^T$$

$$S = \frac{1}{3} \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 30 & 18 & 0 \\ 18 & 24 & -24 \\ 0 & -24 & 96 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

variance of x

$$S = \text{covariance matrix} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

S_{ij} $\forall\, i \neq j$ is called covariance of x_i and x_j

app: correlation b/w sets of vectors

Q13. Find covariance matrix for the given dataset

$$a_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 8 \\ 4 \\ 26 \end{bmatrix} \quad a_3 = \begin{bmatrix} 14 \\ 16 \\ 2 \end{bmatrix} \quad a_4 = \begin{bmatrix} 16 \\ 8 \\ 10 \end{bmatrix}$$

$$M = \frac{1}{4}\begin{bmatrix} 2+8+14+16 \\ 4+4+16+8 \\ 2+26+2+10 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 40 \\ 32 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 2-10 & 8-10 & 14-10 & 16-10 \\ 4-8 & 4-8 & 16-8 & 8-8 \\ 2-10 & 26-10 & 2-10 & 10-10 \end{bmatrix}$$

$$B = \begin{bmatrix} -8 & -2 & 4 & 6 \\ -4 & -4 & 8 & 0 \\ -8 & 16 & -8 & 0 \end{bmatrix} \quad B^T = \begin{bmatrix} -8 & -4 & -8 \\ 2 & -4 & 16 \\ 4 & 8 & -8 \\ 6 & 0 & 0 \end{bmatrix}$$

$$S = \frac{1}{3}\begin{bmatrix} -8 & -2 & 4 & 6 \\ -4 & -4 & 8 & 0 \\ -8 & 16 & -8 & 0 \end{bmatrix}\begin{bmatrix} -8 & -4 & -8 \\ 2 & -4 & 16 \\ 4 & 8 & -8 \\ 6 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 120 & 72 & 0 \\ 72 & 96 & -96 \\ 0 & -96 & 384 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 24 & 0 \\ 24 & 32 & -32 \\ 0 & -32 & 128 \end{bmatrix}$$

Q14. Find covariance matrix for the vectors

$$x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$M = \frac{1}{2}\begin{bmatrix} 3+7 \\ 2+4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3-5 & 7-5 \\ 2-3 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \quad B^T = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$S = \frac{1}{1}\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+4 & 2+2 \\ 2+2 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$

Q15. The following table lists weights and heights of 5 boys

Boy	#1	#2	#3	#4	#5
Weight (lbs)	120	125	125	135	145
Height (in)	61	60	64	68	72

Find covariance matrix of the following data

Sample mean

$$= \frac{1}{5}\left(\begin{bmatrix} 120 \\ 61 \end{bmatrix} + \begin{bmatrix} 125 \\ 60 \end{bmatrix} + \begin{bmatrix} 125 \\ 64 \end{bmatrix} + \begin{bmatrix} 135 \\ 68 \end{bmatrix} + \begin{bmatrix} 145 \\ 72 \end{bmatrix} \right)$$

$$= \frac{1}{5} \begin{bmatrix} 650 \\ 325 \end{bmatrix} = \begin{bmatrix} 130 \\ 65 \end{bmatrix}$$

$$B = \begin{bmatrix} 120-130 & 125-130 & 125-130 & 135-130 & 145-130 \\ 61-65 & 60-65 & 64-65 & 68-65 & 72-65 \end{bmatrix}$$

$$B = \begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix} \qquad B^T = \begin{bmatrix} -10 & -4 \\ -5 & -5 \\ -5 & -1 \\ 5 & 3 \\ 15 & 7 \end{bmatrix}$$

$$S = \frac{1}{4}\begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix}\begin{bmatrix} -10 & -4 \\ -5 & -5 \\ -5 & -1 \\ 5 & 3 \\ 15 & 7 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 100+25+25+25+225 & 40+25+5+15+105 \\ 40+25+5+15+105 & 16+25+1+9+49 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 400 & 190 \\ 190 & 100 \end{bmatrix} = \begin{bmatrix} 100 & 47.5 \\ 47.5 & 25 \end{bmatrix}$$

$$S = \begin{bmatrix} 100 & 47.5 \\ 47.5 & 25 \end{bmatrix}$$

Q16. Construct SVD of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

A is a short matrix

$$A = U \Sigma V^T \Rightarrow AV = U\Sigma$$

$$AA^T = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 333-\lambda & 81 \\ 81 & 117-\lambda \end{vmatrix} = 0$$

$$(\lambda-333)(\lambda-117) - 81^2 = 0$$

$$\lambda^2 - 450\lambda + 32400 = 0$$

$$\lambda_1 = 360 \qquad \lambda_2 = 90$$

eigenvectors

(i) $\lambda_1 = 360$

$$\begin{bmatrix} 333-360 & 81 \\ 81 & 117-360 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -27 & 81 \\ 81 & -243 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} -27 & 81 \\ 0 & 0 \end{bmatrix}$$

$$-27x + 81y = 0$$
$$x = 3y$$

$$u_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

(ii) $\lambda_2 = 90$

$$\begin{bmatrix} 333-90 & 81 \\ 81 & 117-90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 243 & 81 \\ 81 & 27 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/3 R_1} \begin{bmatrix} 243 & 81 \\ 0 & 0 \end{bmatrix}$$

$$243x + 81y = 0$$
$$x = -\frac{1}{3}y$$

$$u_2 = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$U = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$v_1 = \frac{A^T u_1}{\sqrt{360}} = \frac{\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}{\sqrt{360}} = \frac{\begin{bmatrix} 2\sqrt{10} \\ 4\sqrt{10} \\ 4\sqrt{10} \end{bmatrix}}{6\sqrt{10}} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$v_2 = \frac{A^T u_2}{\sqrt{90}} = \frac{\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}}{3\sqrt{10}} = \frac{\begin{bmatrix} 2\sqrt{10} \\ \sqrt{10} \\ -2\sqrt{10} \end{bmatrix}}{3\sqrt{10}} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

To find v_3:

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1}$$

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & -1 & -2 \end{bmatrix}$$

$$-y - 2z = 0 \qquad x - 4z + 2z = 0$$
$$y = -2z \qquad x = 2z$$

$$x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \quad V^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$U = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \quad \Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}}_{V^T}$$

Q17. Compute SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

A is tall matrix

$$A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 18\lambda = 0$$

$$\lambda = 0 \qquad \lambda = 18$$

eigenvectors

i) $\lambda = 18$

$$\begin{bmatrix} 9-18 & -9 \\ -9 & 9-18 \end{bmatrix} = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -9 & -9 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = k \Rightarrow -9x_1 - 9k = 0 \Rightarrow x_1 = -k$

$$x = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = v_1$$

(ii) $\lambda = 0$

$$\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix}$$

Let $x_2 = k \Rightarrow 9x_1 - 9k = 0 \Rightarrow x_1 = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = v_2$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad V^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{0} = 0$$

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A v_1 = u_1 \sigma_1 \qquad \text{or} \qquad u = \frac{A v_1}{\sigma_1}$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = 3\sqrt{2}\, U_1$$

$$U_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

find U_2 & U_3 through orthogonality

$$-\frac{1}{\cancel{3}}x + \frac{2}{\cancel{3}}y - \frac{2}{\cancel{3}}z = 0 \quad \Rightarrow \quad x = \begin{bmatrix} 2y-2z \\ y \\ z \end{bmatrix}$$

$$NS = \left\{ y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Using G·S process

$$q_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - q_1^T \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} q_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$c = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$- \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{5}} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$V^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Q18. Compute $x^T A x$ for the following

(a) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

(a) $x^T A x$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x \\ 3y \end{bmatrix}$$

$$= 4x^2 + 3y^2$$

(b) $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3x - 2y \\ -2x + 7y \end{bmatrix}$$

$$= 3x^2 - 2xy - 2xy + 7y^2$$

$$= 3x^2 + 7y^2 - 4xy$$

Q19. For x in R^3, Let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

Write quadratic form as $x^T A x$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2f x_2x_3 + 2g x_3x_1$$

$a = 5$ $\quad 2h = -1$ $\quad 2f = 8$ $\quad g = 0$

$b = 3$ $\quad h = -1/2$ $\quad f = 4$

$c = 2$

$$A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

Q20. Let $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$.
Check if A is positive definite.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$a = 3$ $\quad h = 2$

$b = 2$ $\quad f = 2$

$c = 1$ $\quad g = 0$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(i) pivots

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2/3 R_1} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2/3 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2/3 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

$-ve$

$\therefore$ not positive definite

Q21. Let $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$

Compute $Q(x)$ for

i) $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ ii) $x = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ iii) $x = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

i) $Q(x) = (-3)^2 - 8(-3)(1) - 5(1)^2$

$= 9 + 24 - 5 = 28$

(ii) $Q(x) = (2)^2 - 8(2)(-2) - 5(-2)^2$

$= 4 + 32 - 20 = 16$

(iii) $Q(x) = (1)^2 - 8(1)(-3) - 5(-3)^2$

$= 1 + 24 - 45 = -20$

Q22. Compute $Q(x)$ form x^TAx for

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (b) $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ (c) $x = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

(a) $$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1+3x_2 & 3x_1+2x_2+x_3 & x_2+x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 4x_1^2 + 3x_1x_2 + 3x_1x_2 + 2x_2^2 + x_2x_3 + x_2x_3 + x_3^2$$

$$= 4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$$

(b) $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

$$Q(x) = 4(4) + 2(1) + 25 + 6(-2) + 2(-5)$$

$$= 16 + 2 + 25 - 12 - 10$$

$$= 21$$

(c) $x = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$Q(x) = 4(1/3) + 2(1/3) + (1/3) + 6(1/3) + 2(1/3)$$

$$= 5$$

Q23. Find SVD of the following

(i) $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$A = U \Sigma V^T$$

(i) $A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 16+9 & 16-9 \\ 16-9 & 16+9 \end{bmatrix}$

$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

$|A^TA - \lambda I| = 0$

$\begin{vmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{vmatrix} = 0$

$\lambda^2 - 50\lambda + 625 - 49 = 0$

$\lambda^2 - 50\lambda + 576 = 0$

$\lambda_1 = 32 \qquad \lambda_2 = 18$ descending order

eigenvectors

(a) $(A - 32I)x = 0$

$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -7 & 7 \\ 0 & 0 \end{bmatrix}$

Let $x_2 = k$

$-7x_1 + 7k = 0 \Rightarrow x_1 = k \to x = \left\{ k\begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$

$$\hat{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(b) $(A - 18I)x = 0$

$$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 7 & 7 \\ 0 & 0 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$\sigma_1 = \sqrt{32} = 4\sqrt{2}$ $\sigma_2 = \sqrt{18} = 3\sqrt{2}$

$$AV = U\Sigma$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$$

$$U_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_2 = \frac{1}{\sigma_2} AV_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$ $V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

A is tall matrix

$$A^T A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$\lambda = 2 \quad \lambda = 3$

eigenvectors

(a) $\lambda = 3$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} -x_1 + 0 &= 0 \\ x_1 &= 0 \end{aligned} \qquad \Rightarrow x = \left\{ k \begin{bmatrix} 0 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(b) $\lambda = 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_2 = 0 \quad \Rightarrow \quad x = \left\{ k \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \sqrt{3} \qquad \sigma_2 = \sqrt{2}$$

$$AV = U\Sigma$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$-y - 2z = 0 \qquad\qquad x - 2z + z = 0$$
$$y = -2z \qquad\qquad x = z$$

$$x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Q24. check if orthogonally diagonalisable

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$A = UDU^T$

yes, $\because$ it is symmetric

characteristic equation

$$-\lambda^3 + 17\lambda^2 - (29 + 29 + 32)\lambda + 144 = 0$$

$$-\lambda^3 + 17\lambda^2 - 90\lambda + 144 = 0$$

$\lambda_1 = 3 \qquad \lambda_2 = 8 \qquad \lambda_3 = 6$

eigenvectors

(a) $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 1/2 R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -2 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -5/2 \end{bmatrix}$$

$$-\frac{5}{2} z = 0 \Rightarrow \quad z = 0$$

$$-2x - 2k = 0$$
$$x = -k$$

$$x = \left\{ k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \; k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

(b) $\lambda = 6$

$$\begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 1/2 R_1} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & -1 & -1/2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 1/2 R_2} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2y - k = 0 \quad \Rightarrow \quad y = -\frac{1}{2} k$$

$$-2x - k = 0 \Rightarrow x = \frac{-1}{2}k$$

$$x = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

3	-2	4
-2	6	2
4	2	3

(c) $\lambda = 3$

$$\begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

null space: $x = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ k \in R \right\}$

$$\hat{x} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$S = U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$S^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$SDS^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$